0/1-Form and 2-Group Symmetries via Boundary Geometries

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Motivation

- Geometric Engineering:

$$\text{SQFT} \hookrightarrow \underset{\text{(Branes, Singularities,...)}}{\text{String Theory}} \twoheadrightarrow \text{SQFT}$$

- Correspondences/Dictionary:

$$\text{Operators, Symmetries,...} \quad \leftrightarrow \quad \text{Geometry } _{\text{(Topology, Diff, Riemannian,...)}}$$

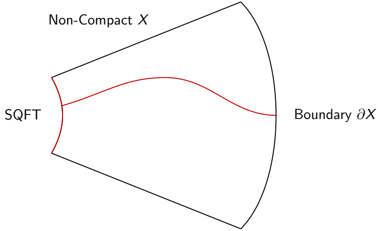
- GKSW: Global Symmetry $\rightarrow$ Topological Defects
- Today's focus: Higher-Symmetries & Topological Structures

[Heckman, Lawrie, Lin, Zhang, Zoccarato, 2022], [Del Zotto, García Etxebarria, Schäfer-Nameki, 2022], [Del Zotto, Heckman, Meynet, Moscrop, Zhang, 2022], [Bhardwaj, Giacomelli, Hübner, Schäfer-Nameki, 2021], [Apruzzi, Bhardwaj, Oh, Schafer-Nameki, 2021], [Apruzzi, Bhardwaj, Gould, Schäfer-Namek, 2021], [Apruzzi, Dierigl, Lin, 2020], [Morrison, Schäfer-Nameki, Willett, 2020], [Del Zotto, Ohmori, 2020], [Albertini, Del Zotto, García Etxebarria, Hosseini, 2020], [Cvetič, Dierigl, Lin, Zhang, 2021], [Del Zotto, Heckman, Park, Rudelius, 2015], ...

- This Presentation:

M-theory + Singularities $\leftrightarrow$ 7d, 5d, 4d SQFTs
Topological Data $\leftrightarrow$ 0/1-form, 2-group Symmetries

- Generic Geometric Set-Up:

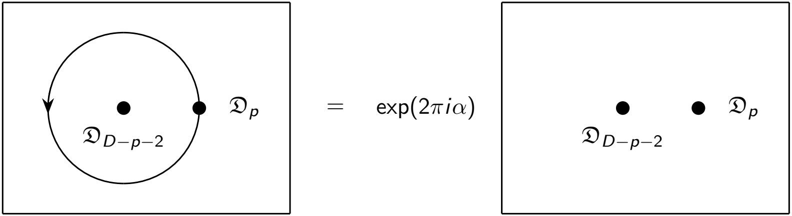


- Geometries X: Elliptic CY_3, Toric $\mathbb{C}^3/\Gamma$, G_2-Spaces

Overview

Defect Group (Field Theory)$_D$

- Defect Group: $\mathfrak{D} = \oplus_p \mathfrak{D}_p$
- Phase ambiguity in correlation functions [Seiberg, Taylor, 2011]



with $\alpha = \langle \mathfrak{D}_p, \mathfrak{D}_{D-p-2} \rangle$.

- Polarizations $\Lambda \subset \mathfrak{D}$ determine absolute theories [Gaiotto, Moore, Neitzke, 2010], [Aharony, Seiberg, Tachikawa, 2013], [Gukov, Hsin, Pei, 2020]
- The higher symmetries are then the Pontryagin dual $\Lambda^\vee$

Defect Group (M-theory on X)

- $\mathcal{D}_p = \mathcal{D}_p^{\text{M2}} \oplus \mathcal{D}_p^{\text{M5}}$
- M2, M5 on relative cycles [Morrison, Schäfer-Nameki, Willett, 2020], [Albertini, Del Zotto, García Etxebarria, Hosseini, 2020], [Del Zotto, Heckman, Park, Rudelius, 2015]

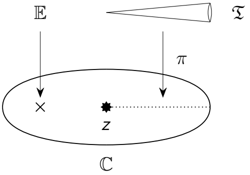
$$\mathcal{D}_p^{\text{M2}} = \text{Tor}\,\frac{H_{3-p}(X,\partial X)}{H_{3-p}(X)} \cong \text{Tor}\, H_{3-p-1}(\partial X)|_{\text{triv}}$$

$$\mathcal{D}_p^{\text{M5}} = \text{Tor}\,\frac{H_{6-p}(X,\partial X)}{H_{6-p}(X)} \cong \text{Tor}\, H_{6-p-1}(\partial X)|_{\text{triv}}$$

- $\langle \cdot, \cdot \rangle \;\leftrightarrow\;$ Linking Pairing $\ell(\,\cdot\,,\,\cdot\,)$ in ∂X

Example: M-theory on Local K3s

- Local K3: $X \to \mathbb{C}$ with singularity of Kodaira type Φ at $z \in \mathbb{C}$



- Boundary $\partial X \to S^1$ with monodromy M_1, we use

$$0 \to \text{coker}\,(M_n - 1) \to H_n(\partial X) \to \ker\,(M_{n-1} - 1) \to 0$$

- $\mathcal{D}_1^{\text{M2}} = \mathcal{D}_4^{\text{M5}} = \text{Tor}\, H_2(X, \partial X)/H_2(X) \cong \text{Tor}\,\text{Coker}(M_1 - 1) = \langle \mathfrak{T} \rangle$
- X engineers 7d SYM with gauge algebra $\mathfrak{g}_\Phi$
- Defect group $\mathcal{D} = \langle \mathfrak{T} \rangle_1^{\text{M2}} \oplus \langle \mathfrak{T} \rangle_4^{\text{M5}}$

(Local K3s Continued)

- We have determined the defect group, now determine maximally mutually local subgroups
- Resolve Kodaira Singularity $\widetilde{X} \to X$, exceptional curves C_{α_i}
- Dualize to linear forms via intersection pairing

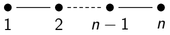
$$
\begin{aligned}
\alpha : H_2(\widetilde{X}) \to H_2(\widetilde{X})^* \,, &\qquad C_{\alpha_i} \mapsto (\, C_{\alpha_i}, \, \cdot \,)\,, \\
\beta : H_2(\widetilde{X}, \partial X) \to H_2(\widetilde{X})^* \,, &\qquad \widehat{\mathfrak{T}} \mapsto (\widehat{\mathfrak{T}}, \, \cdot \,)
\end{aligned}
$$

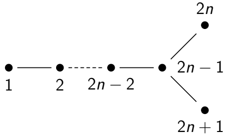
where $\mathrm{Im}(\alpha) \subset \mathrm{Im}(\beta)$.

- Thimbles $\mathfrak{T}$ admit compact representatives in $H_2(\widetilde{X}, \mathbb{Q}/\mathbb{Z})$

(Local K3s Continued)

Example of compact representatives for Kodaira Thimbles for singularities of Kodaira type $\Phi = I_n, I^*_{2n-3}$ $(\mathfrak{g} = \mathfrak{su}, \mathfrak{so})$





$$\mathfrak{T}_{I_n} = \frac{1}{n} \sum_{i=1}^{n-1} i C_{\alpha_i}$$

$$\mathfrak{T}_{I_n} \cdot \mathfrak{T}_{I_n} = 1/n$$

$$\mathfrak{T}_{I^*_{2n-3}} = \frac{1}{4} C_{\alpha_{2n+1}} + \frac{3}{4} C_{\alpha_{2n}} + \frac{1}{2} \sum_{i=1}^{n} C_{\alpha_{2i-1}}$$

$$\mathfrak{T}_{I^*_{2n-3}} \cdot \mathfrak{T}_{I^*_{2n-3}} = 1/4, 3/4 \quad (n = \text{odd,even})$$

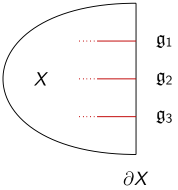
$\Rightarrow$ Talk at Freiburg Simon's Meeting 8th June

Example (Continued)

- Non-trivial self-linking/intersection $\ell(\partial \mathfrak{T}, \partial \mathfrak{T}) = \mathfrak{T} \cdot \mathfrak{T} \neq 0$
- Elements of $\mathcal{D}_1^{\mathrm{M2}}, \mathcal{D}_4^{\mathrm{M5}}$ generically mutually non-local
- Choose electric polarization $\mathcal{D}_1^{\mathrm{M2}}$ (throughout this talk)
- Gauge group is simply connected G_Φ with algebra $\mathfrak{g}_\Phi$
- Engineered 7d SYM theory with gauge group G_Φ
- Wilson line operators $\mathcal{D}_1^{\mathrm{M2}}$ acted on by 1-form symmetry Z_{G_Φ}
- This example generalizes straightforwardly.

0-Form Flavor Symmetries

- When are flavor symmetries present?
- Non-compact ADE loci (= flavor branes) → Flavor symmetries
- Naive (non-abelian) flavor symmetry $\widetilde{G}_F$:

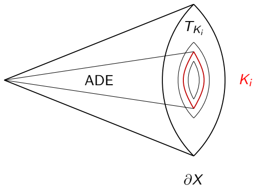


$$\widetilde{G}_F = \widetilde{G}_1 \times \widetilde{G}_2 \times \widetilde{G}_3 \times \dots$$

$\widetilde{G}_i$ simply connected with algebra $\mathfrak{g}_i$

- Flavor Wilson lines → global form of flavor symmetry
- Gauge Wilson lines: non-compact two-cycles in X fibered by vanishing cycles
- Flavor Wilson lines: compact two-cycles in ∂X fibered by vanishing cycles

- Define $K = \cup_i K_i$ as ADE locus in boundary ∂X
- Define the tube T_K and smooth boundary $\partial X^\circ = \partial X \setminus K$
- Locally $T_K \cap \partial X^\circ \cong \cup_i K_i \times S^3/\Gamma_i$



- ADE: $\mathrm{Tor}\, H_1(S^3/\Gamma_i) = Z_{\widetilde{G}_i}$
- Naive Flavor Center

$$Z_{\widetilde{G}_F} = \mathrm{Tor}\, H_1(T_K \cap \partial X^\circ) \cong Z_{\widetilde{G}_1} \oplus Z_{\widetilde{G}_2} \oplus Z_{\widetilde{G}_3} \oplus \dots$$

Flavor Wilson Lines

- Mayer-Vietoris sequence for covering $\partial X = \partial X^\circ \cup T_K$

$$\dots \to H_n(\partial X^\circ \cap T_K) \xrightarrow{\iota_n} H_n(\partial X^\circ) \oplus H_n(T_K) \to H_n(\partial X) \xrightarrow{\partial_n} \dots$$

- Flavor Wilson lines

$$\begin{aligned} Z_{G_F} &\cong \mathrm{Tor}\,\mathrm{Im}\left(\partial_2 : H_2(\partial X) \to H_1(\partial X^\circ \cap T_K) \cong Z_{\widetilde{G}_F}\right) \\ &= \mathrm{Tor}\,\mathrm{Ker}\left(\iota_1 : Z_{\widetilde{G}_F} \cong H_1(\partial X^\circ \cap T_K) \to H_1(\partial X^\circ) \oplus H_1(T_K)\right) \end{aligned}$$

$\equiv$ two-cycles fibered by vanishing one-cycles of the ADE singularities

Example: 5d T_N Theory

- $X = \mathbb{C}^3/\mathbb{Z}_N \times \mathbb{Z}_N$ where ($\omega^N = \eta^N = 1$)

$$(z_1, z_2, z_3) \sim (\omega z_1, \eta z_2, (\omega\eta)^{-1} z_3)$$

  Three A_{N-1} planes $z_i = z_j = 0$. Trivial 1-form symmetry [Tian, Wang, 2021], [Del Zotto, Heckman, Meynet, Moscrop, Zhang, 2022]
- Flavor algebra $\mathfrak{su}(N)^3$
- Toric coordinates: $p_i = |z_i|^2$ and $\theta_i = \arg z_i$, three circle worths of A_{N-1} singularities in ∂X

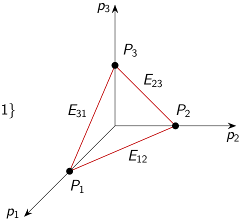
$T^3/\mathbb{Z}_N^2 \to \partial X \to \Delta$

Torus $T^3 = (\theta_1, \theta_2, \theta_3)$

Boundary $\partial X = \{p_1 + p_2 + p_3 = 1\}$

Edges E_{jk}

Corners P_i

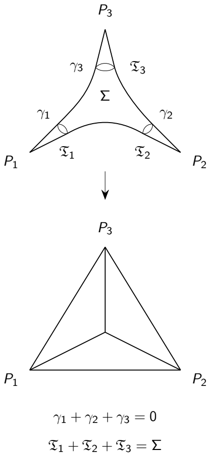


(T_N Continued)

- Trivial $T^3 = S^1_1 \times S^1_2 \times S^1_3$ fibration, following one-cycles collapse at ADE singularities

$$\begin{aligned} P_1 &: \quad \gamma_1 = (S^1_2 - S^1_3)/N \\ P_2 &: \quad \gamma_2 = (S^1_3 - S^1_1)/N \\ P_3 &: \quad \gamma_3 = (S^1_1 - S^1_2)/N \end{aligned}$$

- Relative two-cycles $\mathfrak{T}_i$ fibered by γ_i glue to two-cycle Σ



(T_N Continued)

- Torsional two-cycles $H_2(\partial X) \cong \mathbb{Z}_N$ generated by Σ
- Diagonal embedding

$$\partial_2 : \mathbb{Z}_N \cong H_2(\partial X) \rightarrow H_1(\partial X^\circ \cap T_K) \cong \mathbb{Z}_N^3$$

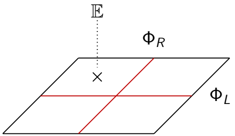
- Flavor Symmetry and Center [Bhardwaj, 2021]

$$G_F = SU(N)^3/\mathbb{Z}_N \times \mathbb{Z}_N \,, \qquad Z_{G_F} \cong \mathrm{Im}\, \partial_2 \cong \mathbb{Z}_N$$

- $T_3 = E_6$ Minahan-Nemeschansky, $G_F = SU(3)^3/\mathbb{Z}_3^2$ is compatible with enhancement to $G_F = E_6/\mathbb{Z}_3$. [Bhardwaj, 2021]

Example: $(G_{\mathrm{ADE}}, G_{\mathrm{ADE}})$ Conformal Matter

- Elliptic three-fold $\mathbb{E} \hookrightarrow X_3 \to B = \mathbb{C}^2$ [Del Zotto, Heckman, Tomasiello, Vafa, 2014]
- Discriminant Locus Φ_L on $\mathbb{C} \times \{0\}$ and Φ_R on $\{0\} \times \mathbb{C}$



- Boundary five-manifold $\mathbb{E} \hookrightarrow \partial X_3 \to S^3$ where

$$T^2 = S^1_L \times S^1_R \hookrightarrow S^3 \to [0,1]$$

- Discriminant locus consists of two linking circles in S^3 (Hopf link)

(Example Continued)

- Now characterize the spectrum of two-cycles
- Excise the singular fibers

$$\mathbb{E} \hookrightarrow \partial X^\circ \to S^1_L \times S^1_R$$

- Short exact sequence for spaces $X \to S^1$ fibered over circles

$$0 \to \text{coker}\,(M_n - 1) \to H_n(X) \to \ker\,(M_{n-1} - 1) \to 0$$

where M_n is the monodromy in homology in degree n.

- Monodromies M_{Φ_L}, M_{Φ_R} about S^1_L, S^1_R respectively, it follows

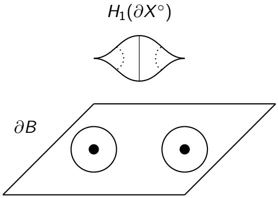
$$\text{Tor}\, H_1(\partial X^\circ) = \text{Tor}\, \frac{\mathbb{Z}^2}{\text{Im}(M_{\Phi_L} - 1) \cup \text{Im}(M_{\Phi_R} - 1)}$$

(Example Continued)

- Here Tor $H_1(\partial X^\circ)$ characterized one-cycles which collapse at both discriminant components
- $(SU(n), SU(m))$ Conformal Matter (=Bifundamental Matter)

$$G_F = \frac{SU(n) \times SU(m)}{\mathbb{Z}_{\text{gcd}(n,m)}}$$

- More generally $G_F = G_L \times G_R / Z_{\text{diag}}$



2-Groups

- Two key short exact sequence (and Postnikov class) [Lee, Ohmori, Tachikawa, 2021], [Benini, Cordova, Hsin, 2019], ...

$$
\begin{aligned}
0 \to \mathcal{C} \to Z_{\widetilde{G}_F} \to Z_{G_F} \to 0 \\
0 \to \mathcal{C}^\vee \to \widetilde{\mathcal{A}}^\vee \to \mathcal{A}^\vee \to 0
\end{aligned}
$$

- Z_{G_F} : Center Flavor Symmetry
- $Z_{\widetilde{G}_F}$: Naive Center Flavor Symmetry
- $\mathcal{A}^\vee$: Line Operators modulo screening by local operators
- $\widetilde{\mathcal{A}}^\vee$: Line Operators modulo screening by local operators transforming in reps of Z_{G_F}
- $\mathcal{C}^\vee$: Line Operators in the kernel of $\widetilde{\mathcal{A}}^\vee \to \mathcal{A}^\vee$

Orbifold Homology

$$0 \to \mathcal{C}^\vee \to \widetilde{\mathcal{A}}^\vee \to \mathcal{A}^\vee \to 0$$

- Equivariant Case: Global quotient $X = Y/\Gamma$, Y contractible

  $\widetilde{\mathcal{A}}^\vee$: M2 branes wrapped on $H_1^{\text{equiv}}(\partial X)$
- Short exact sequence (projection onto singular homology):

$$0 \to \ker p \to H_1^{\mathrm{equiv}}(\partial X) \xrightarrow{p} H_1(\partial X) \to 0$$

- Identifications:

$$\mathcal{A}^\vee = H_1(\partial X) \qquad \text{(line operators/defects)}$$
$$\mathcal{C}^\vee = \ker p \qquad \text{(twisted sector)}$$

- General Case: Equivariant Homology $\to$ Orbifold Homology

  $\widetilde{\mathcal{A}}^\vee$: M2 branes wrapped on $H_1^{\text{orb}}(\partial X)$

Codimension-4 ADE Singularities

$$0 \to \mathcal{C}^\vee \to \widetilde{\mathcal{A}}^\vee \to \mathcal{A}^\vee \to 0$$

$$0 \to H_1^{\text{twist}}(\partial X) \to H_1^{\text{orb}}(\partial X) \xrightarrow{p} H_1(\partial X) \to 0$$

- Characterization in singular homology [Thurston, 1980], [Moerdijk, Pronk, 1997]

$$H_1^{\text{orb}}(\partial X) \cong H_1(\partial X^\circ)$$

where $\partial X^\circ = \partial X \setminus K$ with ADE locus K.

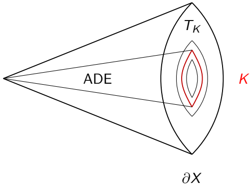
- But we encountered $H_1(\partial X^\circ)$ earlier already...

2-groups and Mayer-Vietoris

- Mayer-Vietoris sequence for covering $\partial X = \partial X^\circ \cup T_K$

$$\ldots \to H_n(\partial X^\circ \cap T_K) \xrightarrow{\iota_n} H_n(\partial X^\circ) \oplus H_n(T_K) \to H_n(\partial X) \xrightarrow{\partial_n} \ldots$$

- Tube T_K deformation retracts to ADE locus K
- ADE locus K has simple topology (eg. circles in 5d examples)



- We derive the exact sequence

$$0 \to \ker(\iota_1) \to H_1\big(\partial X^\circ \cap T_K\big) \xrightarrow{\iota_1} H_1\big(\partial X^\circ\big) \oplus H_1(T_K) \to H_1(\partial X) \to 0\,.$$

- Which maps (after removing trivial free factors and reversing arrows) to the symmetry relations

$$0 \to \mathcal{A} \to \widetilde{\mathcal{A}} \to Z_{\widetilde{G}_F} \to Z_{G_F} \to 0\,.$$

- By general properties of exact sequences we have the split

$$\begin{array}{ccccccccc} 0 & \rightarrow & \mathcal{C} & \rightarrow & Z_{\widetilde{G}_F} & \rightarrow & Z_{G_F} & \rightarrow & 0 \\ 0 & \rightarrow & \mathcal{C}^\vee & \rightarrow & \widetilde{\mathcal{A}}^\vee & \rightarrow & \mathcal{A}^\vee & \rightarrow & 0 \end{array}$$

- Which is contained in the geometry as

$$0 \rightarrow \ker(\iota_1) \rightarrow H_1\big(\partial X^\circ \cap T_K\big) \xrightarrow{\iota_1} \frac{H_1\big(\partial X^\circ \cap T_K\big)}{\ker(\iota_1)} \rightarrow 0\,,$$

$$0 \rightarrow \frac{H_1\big(\partial X^\circ \cap T_K\big)}{\ker(\iota_1)} \rightarrow H_1\big(\partial X^\circ\big) \oplus H_1(T_K) \rightarrow H_1(\partial X) \rightarrow 0\,.$$

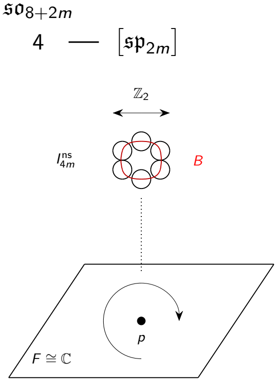
- Postnikov class is the Bockstein of the extension class for the SES characterizing the global form of the flavor symmetry
- $\Rightarrow$ 0-form, 1-form, 2-group symmetries from cutting and gluing of orbifolds

Example: 5d $Spin(8+2m)$ with $2m$ Vectors

- Elliptic $X_3 \to B$, $B = \mathcal{O}_{\mathbb{P}^1}(-4)$
- Discriminant Locus

$$\begin{aligned} \mathbb{P}^1 &: \ I_m^{*,\mathrm{s}} \\ F \subset \mathcal{O}_{\mathbb{P}^1}(-4) &: \ I_{4m}^{\mathrm{ns}} \end{aligned}$$

- (n)s = (non)-split
- At Ramification point p one-cycle B collapses



(Example Continued)

$$\mathbb{E} \hookrightarrow \partial X_3 \rightarrow S^3/\mathbb{Z}_4 = \partial\mathcal{O}_{\mathbb{P}^1}(-4)$$

- Tor $H_1(\partial X) \cong \mathbb{Z}_4 \oplus \mathbb{Z}_2$: Hopf fiber of the base $S_3/\mathbb{Z}_4$ and B
- Tor $H_1(\partial X^\circ)$: Excising singular fibers, implies for base

$$S^1 \hookrightarrow S^3/\mathbb{Z}_4 \rightarrow S^2 \setminus \{*\}$$

- Now $S^2 \setminus \{*\}$ deformation retracts to a point
- Base $(S^3/\mathbb{Z}_4) \setminus S^1_H$ deformation retracts to Hopf fiber $(S^1_H)'$
- ∂X° deformation retracts to three-manifold $\mathbb{E} \hookrightarrow M_3 \rightarrow (S^1_H)'$

(Example Continued)

- The Hopf circle $(S^1_H)'$ links both S^1_H and the bulk $\mathbb{P}^1$, their monodromies are

$$M_{I_m^*} = \begin{pmatrix} -1 & -m \\ 0 & -1 \end{pmatrix}, \qquad M_{I_{4m}} = \begin{pmatrix} 1 & 4m \\ 0 & 1 \end{pmatrix}$$

- Therefore ∂X° deformation retracts to three-manifold $\mathbb{E} \hookrightarrow M_3 \to (S^1_H)'$ with monodromy

$$M = \begin{pmatrix} -1 & -5m \\ 0 & -1 \end{pmatrix}.$$

- We conclude

$$\text{Tor}\, H_1(\partial X^\circ) = \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4\,, & m \in 2\mathbb{Z} \\ \mathbb{Z}_4 \oplus \mathbb{Z}_4\,, & m \in 2\mathbb{Z}+1 \end{cases}$$

(Example Continued)

$$0 \to \mathcal{C} \to Z_{\widetilde{G}_F} \to Z_{G_F} \to 0$$
$$0 \to \mathcal{C}^\vee \to \widetilde{\mathcal{A}}^\vee \to \mathcal{A}^\vee \to 0$$

For odd m we have:

$$0 \to \mathbb{Z}_2 \to \mathbb{Z}_2 \to 0 \to 0$$
$$0 \to \mathbb{Z}_2 \to \mathbb{Z}_4 \oplus \mathbb{Z}_4 \to \mathbb{Z}_2 \oplus \mathbb{Z}_4 \to 0$$

For even m we have:

$$0 \to \mathbb{Z}_2 \to \mathbb{Z}_2 \to 0 \to 0$$
$$0 \to \mathbb{Z}_2 \to \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4 \to \mathbb{Z}_2 \oplus \mathbb{Z}_4 \to 0$$

The flavor symmetry is $G = Sp(2m)/\mathbb{Z}_2$ [Apruzzi, Bhardwaj, Oh, Schafer-Nameki, 2021]

When m odd we have a non-trivial 2-group symmetry.

Conclusion and Omissions

- We considered SQFTs geometrically engineered in M-theory
- Geometry boundaries contained ADE singularities
- Motivated by Orbifold Homology we gave a prescription in singular homology to compute the 0-form, 1-form and 2-group symmetries of the SQFT

- In 2203.10022 we systematically study the non-compact cycles of elliptic threefolds and compute anomalies for 1-form symmetries via triple intersections in geometry
- In 2203.10102 we further analyze G_2 spaces constructed topologically as uplifts of D6 brane configurations

Outlook

- Cutting and Gluing for global models
- Anomalies via differential orbifold homology and the formalism of symmetry TFTs
- n-groups