Getting High on Gluing Orbifolds (Part 2)

Max Hübner

Penn
University of Pennsylvania

2203.10022 with David R. Morrison, Sakura Schäfer-Nameki and Yi-Nan Wang
2203.10102 with Mirjam Cvetič, Jonathan J. Heckman, Ethan Torres

University of Freiburg

Geometry, Topology and Singular Special Holonomy Spaces

June 8$^{\text{th}}$ 2022

Overview

Elliptic Calabi-Yau Threefolds

- Non-compact singular elliptically fibered Calabi-Yau threefolds $\pi : X \to B$ with section $\sigma : B \to X$ and discriminant locus Δ
- 5d SQFT engineered by M-theory on X
- 5d line defects: M2 branes on relative cycles

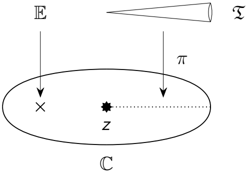
$$\mathfrak{h}_{(2)} = \text{Tor}\,\frac{H_2(X,\partial X)}{H_2(X)} \cong \text{Tor}\, H_1(\partial X)\big|_{\text{triv.}}$$

- Step 1: Compute $\mathfrak{h}_{(2)}$, two contributions
  - Cycles of the base B lifted to X via the section σ, $\mathfrak{h}_{(2,B)}$
  - Cycles with one leg in B fibered by vanishing cycle in $\mathbb{E}$, $\mathfrak{h}_{(2,F)}$
- Latter contain gauge theory data, we therefore focus on $\mathfrak{h}_{(2,F)}$

Kodaira Thimbles

We take a step back an consider local K3s, model for the geometry normal to Δ

- Local K3: $X \to \mathbb{C}$ with singularity of Kodaira type Φ at $z \in \mathbb{C}$



- Boundary $\partial X \to S^1$ with monodromy M_1, we use

$$0 \to \text{coker}\,(M_1 - 1) \to H_1(\partial X) \to \ker\,(M_0 - 1) \to 0$$

- $\mathfrak{h}_{(2)} = \text{Tor}\, H_2(X, \partial X)/H_2(X) \cong \text{Tor}\,\text{Coker}(M_1 - 1) = \langle \mathfrak{T} \rangle$

Compact Presentation of Kodaira Thimbles

- Resolve Kodaira Singularity $\widetilde{X} \to X$, exceptional curves C_{α_i}
- Dualize to linear forms via intersection pairing

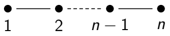
$$\begin{aligned} \alpha : H_2(\widetilde{X}) \to H_2(\widetilde{X})^* \,, \qquad & C_{\alpha_i} \mapsto (\, C_{\alpha_i}, \, \cdot \,) \,, \\ \beta : H_2(\widetilde{X}, \partial X) \to H_2(\widetilde{X})^* \,, \qquad & \widehat{\mathfrak{T}} \mapsto (\widehat{\mathfrak{T}}, \, \cdot \,) \end{aligned}$$

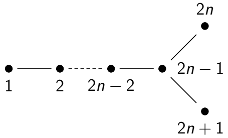
where $\mathrm{Im}(\alpha) \subset \mathrm{Im}(\beta)$ over $\mathbb{Z}$.

- But $\mathrm{Im}(\alpha) = \mathrm{Im}(\beta)$ over $\mathbb{Q}$ and therefore Kodaira Thimbles admit compact representatives in $H_2(\widetilde{X}, \mathbb{Q}/\mathbb{Z})$

Example of Kodaira Thimbles

Example of compact representatives for Kodaira Thimbles for singularities of Kodaira type $\Phi = I_n, I^*_{2n-3}$ $(\mathfrak{g} = \mathfrak{su}, \mathfrak{so})$





$$\mathfrak{T}_{I_n} = \frac{1}{n} \sum_{i=1}^{n-1} i C_{\alpha_i}$$

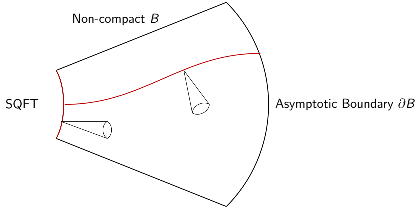
$$\mathfrak{T}_{I_n} \cdot \mathfrak{T}_{I_n} = 1/n$$

$$\mathfrak{T}_{I^*_{2n-3}} = \frac{1}{4} C_{\alpha_{2n+1}} + \frac{3}{4} C_{\alpha_{2n}} + \frac{1}{2} \sum_{i=1}^{n} C_{\alpha_{2i-1}}$$

$$\mathfrak{T}_{I^*_{2n-3}} \cdot \mathfrak{T}_{I^*_{2n-3}} = 1/4, 3/4 \quad (n = \text{odd,even})$$

Threefolds

- Back to: Non-compact elliptic threefold $\pi : X \to B$
- Connected discriminant $\Delta = \cup_i \Delta_i$ with compact and non-compact components



- Resolution $\widetilde{X} \to X$, introduces two types of exceptional curves (in cases considered here)

  (1) Co-dimension-one: Normal bundle $\mathcal{O}(-2) \oplus \mathcal{O}(0)$

  (2) Co-dimension-two: Normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$

Relative Cycles for Threefolds

- Now compute $\mathfrak{h}_{(2)}$ where

$$\mathfrak{h}_{(2)} = \text{Tor}\, \frac{H_2(X, \partial X)}{H_2(X)}$$

by taking the quotient by all curves in $H_2(X)$ not in in class (2).

- Cycles with one leg in base and fiber are then

$$\{\mathfrak{T}_i \,|\, \text{Kodaira Thimble for compact } \Delta_i\}$$

- Now quotient by the matter curves (2), this yields identifications among Kodaira Thimbles

$$\mathfrak{h}_{(2),F} = \{\mathfrak{T}_i \,|\, \text{Kodaira Thimble for compact } \Delta_i\} \,/ \sim_{(2)}$$

Center Divisors

- Dual formulation in terms of divisors
- Center divisors $\mathfrak{D}$ are classes in $H_4(X,\mathbb{Q})$ intersecting all curves integrally and are given by rational linear combinations of Cartan divisors D_{α_i}
- Divisors dual to Kodaira thimbles $\mathfrak{T} = q^i C_{\alpha_i}$ are rational combinations of Cartan divisors $\widehat{\mathfrak{T}} = q^i D_{\alpha_i}$ where $q^i \in \mathbb{Q}/\mathbb{Z}$, they admit the pairing

$$\langle \cdot , \cdot \rangle : \quad H_4(X,\mathbb{Q}/\mathbb{Z}) \times H_2(X,\mathbb{Q}/\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$$

- Constraint of integral intersection with curves of type (2) gives

$$\widehat{\mathfrak{h}}_{(2,F)} = \left\{ \mathfrak{D}_n = \sum_i n_i \widehat{\mathfrak{T}}_i \;\middle|\; \begin{array}{l} \mathfrak{D}_n \text{ has integral intersection with} \\ \text{all matter curves (2), } n_i \in \mathbb{Z} \end{array} \right\}$$

- We have $\widehat{\mathfrak{h}}_{(2,F)} \cong \mathfrak{h}_{(2,F)}$

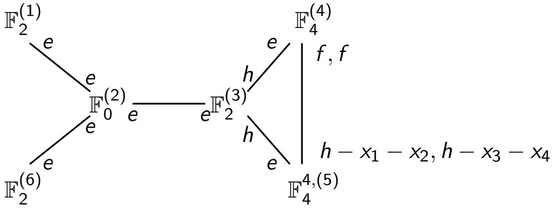
Example: 5d $Spin(10) + 2V$

- Geometry:

$$\overset{\mathfrak{so}(10)}{4} - [\mathfrak{sp}(2)]$$

- Resolved geometry, Divisors (Hirzebruch $\mathbb{F}$), Curves (e, f, h, x_i):

[H, Morrison, Schäfer-Nameki, Wang, 2022]



Label the compact curves as

$$C_1 = e|_{D_2} \ , \ C_2 = f|_{\mathfrak{D}_2} \ , \ C_3 = f|_{D_1} \ , \ C_4 = f|_{D_6} \ , \ C_5 = f|_{D_3} \ , \ C_6 = f|_{D_4} \ ,$$
$$C_7 = x_1|_{D_5} \ \ C_8 = x_2|_{D_5} \ , \ C_9 = x_3|_{D_5} \ , \ C_{10} = x_4|_{D_5} \ .$$

The intersection form $\mathcal{M}_{ij} = D_i \cdot C_j$ is

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}
D_1	0	1	−2	0	0	0	0	0	0	0
D_2	−2	−2	1	1	1	0	0	0	0	0
D_3	0	1	0	0	−2	1	0	0	0	0
D_4	0	0	0	0	1	−2	1	1	1	1
D_5	0	0	0	0	1	0	−1	−1	−1	−1
D_6	0	1	0	−2	0	0	0	0	0	0

The center divisors for $\mathfrak{h}_2 = \mathbb{Z}_2 \times \mathbb{Z}_4$ are

$$\mathfrak{D}_{\mathbb{Z}_2} = \frac{1}{2}(D_1 + D_6) \ , \ D_{\mathbb{Z}_4} = \frac{1}{4}(D_1 + 2D_2 + 2D_3 - D_4 - D_5 + D_6)$$

and here $\mathfrak{h}_{2,F} \cong \mathbb{Z}_2$.

(Example Continued)

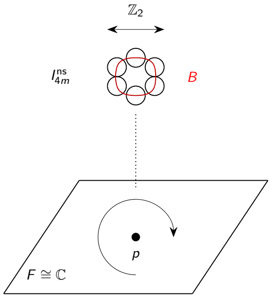
We can identify the thimble generating $\mathfrak{h}_{2,F} \cong \mathbb{Z}_2$ directly in the singular geometry. (m=1).

- Elliptic $X_3 \to B$, $B = \mathcal{O}_{\mathbb{P}^1}(-4)$
- Discriminant Locus

$$\begin{aligned} \mathbb{P}^1 &: \; I_m^{*,\mathrm{s}} \\ F \subset \mathcal{O}_{\mathbb{P}^1}(-4) &: \; I_{4m}^{\mathrm{ns}} \end{aligned}$$

- (n)s = (non)-split
- At Ramification point p one-cycle B collapses
- Tor $H_1(\partial X) \cong \mathbb{Z}_4 \oplus \mathbb{Z}_2$: Hopf fiber in $S_3/\mathbb{Z}_4$ and B

$$\overset{\mathfrak{so}_{8+2m}}{4} \;\text{—}\; [\mathfrak{sp}_{2m}]$$



(Example Continued)

$$\mathbb{E} \hookrightarrow \partial X_3 \rightarrow S^3/\mathbb{Z}_4 = \partial\mathcal{O}_{\mathbb{P}^1}(-4)$$

- Tor $H_1(\partial X) \cong \mathbb{Z}_4 \oplus \mathbb{Z}_2$: Hopf fiber of the base $S_3/\mathbb{Z}_4$ and B
- Excise singular fibers ∂X_F, implies for base

$$S^1 \hookrightarrow S^3/\mathbb{Z}_4 \rightarrow S^2 \setminus \{*\}$$

- Now $S^2 \setminus \{*\}$ deformation retracts to a point
- Base $(S^3/\mathbb{Z}_4) \setminus S^1_H$ deformation retracts to Hopf fiber $(S^1_H)'$
- ∂X_F deformation retracts to three-manifold $\mathbb{E} \hookrightarrow M_3 \rightarrow (S^1_H)'$

(Example Continued)

- The Hopf circle $(S^1_H)'$ links both S^1_H and the bulk $\mathbb{P}^1$ with monodromies

$$M_{I_m^*} = \begin{pmatrix} -1 & -m \\ 0 & -1 \end{pmatrix} , \qquad M_{I_{4m}} = \begin{pmatrix} 1 & 4m \\ 0 & 1 \end{pmatrix}$$

- Retraction of ∂X_F to three-manifold $M_3 \rightarrow (S^1_H)'$ with monodromy

$$M = \begin{pmatrix} -1 & -5m \\ 0 & -1 \end{pmatrix} .$$

- We conclude $(\partial X_F \rightarrow \partial X^\circ)$

$$\text{Tor}\, H_1(\partial X^\circ) = \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4 \,, & m \in 2\mathbb{Z} \\ \mathbb{Z}_4 \oplus \mathbb{Z}_4 \,, & m \in 2\mathbb{Z}+1 \end{cases}$$

(Example Continued)

$$0 \to \mathcal{C} \to Z_{\widetilde{G}_F} \to Z_{G_F} \to 0$$
$$0 \to \mathcal{C}^\vee \to \widetilde{\mathcal{A}}^\vee \to \mathcal{A}^\vee \to 0$$

For odd m we have:

$$0 \to \mathbb{Z}_2 \to \mathbb{Z}_2 \to 0 \to 0$$
$$0 \to \mathbb{Z}_2 \to \mathbb{Z}_4 \oplus \mathbb{Z}_4 \to \mathbb{Z}_2 \oplus \mathbb{Z}_4 \to 0$$

For even m we have:

$$0 \to \mathbb{Z}_2 \to \mathbb{Z}_2 \to 0 \to 0$$
$$0 \to \mathbb{Z}_2 \to \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4 \to \mathbb{Z}_2 \oplus \mathbb{Z}_4 \to 0$$

The flavor symmetry is $G = Sp(2m)/\mathbb{Z}_2$ [Apruzzi, Bhardwaj, Oh, Schafer-Nameki, 2021]

When m odd we have a non-trivial 2-group symmetry.

D6-brane uplifts

IIA Configuration:

- CY_3 X_6 with N_i D6 branes wrapped on sLag submanifold M_i
- D6-branes source RR flux $F_2 = dC_1$ counting branes $\int_{S^2} F_2 = n_{D6}$
- Expand F_2 in fluxes through cycles linking D6 loci

M-theory lift [Acharya, Gukov, 2004]:

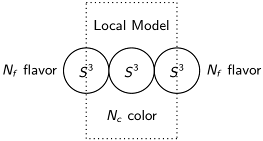
- Local geometry normal to N_i D6 branes lifts to $\mathbb{C}^2/\mathbb{Z}_{N_i}$
- $X_6^\circ = X_6 \setminus \cup_i M_i$ lifts a circle bundle $X_7^\circ \to X_6^\circ$ with Euler class $e = F$
- IIA set-up lifts to a circle bundle $X_7 \to X_6$ with $A_{N_i - 1}$ loci

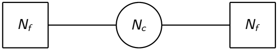
Restrict constructions to the boundary ∂X_7

$$\cdots \to H^k(\partial X_7^\circ) \to H^{k-1}(\partial X_6^\circ) \xrightarrow{e\wedge} H^{k+1}(\partial X_6^\circ) \to H^{k+1}(\partial X_7^\circ) \to \dots .$$

Example

- CY_3 [Feng, He, Kennaway, Vafa, 2008], [Del Zotto, Oh, Zhou, 2021] with supersymmetric three-spheres
- Consider local geometry $X_6 = T^*S^3$ of a fixed (color) three-sphere
- Color S^3 intersects flavor S^3's at points
- Flavor S^3's decompactify to fiber classes topologically $\mathbb{R}^3 \subset T^*S^3$





- We have $\partial X_6 = S^2 \times S^3$ and $\partial X_6^\circ = S^2 \times (S^3 \setminus \{*_1, *_2\}) \sim S^2_F \times S^2_B$

(Example Continued)

- $\partial X_6^\circ = S^2 \times (S^3 \setminus \{*_1, *_2\}) \sim S_F^2 \times S_B^2$
- $2 \times N_f$ flavor D6 branes and N_c color D6 branes
- $F = N_c \, \mathrm{PD}[S_f^2] + N_f \, \mathrm{PD}[S_c^2]$
- Non-trivial subsequence of the Gysin sequence:

$$0 \to H^1(\partial X_7^\circ) \to H^0(S_B^2 \times S_F^2) \xrightarrow{e_0} H^2(S_B^2 \times S_F^2) \to H^2(\partial X_7^\circ) \to 0$$
$$0 \to H^3(\partial X_7^\circ) \to H^2(S_B^2 \times S_F^2) \xrightarrow{e_2} H^4(S_B^2 \times S_F^2) \to H^4(\partial X_7^\circ) \to 0$$

$$e_0 : \mathbb{Z} \to \mathbb{Z}^2 , \quad k \mapsto (kN_f, kN_c) , \qquad e_2 : \mathbb{Z}^2 \to \mathbb{Z} , \quad (n, m) \mapsto nN_f + mN_c \, .$$

(Example Continued)

- Homology groups:

$$H_*(\partial X_7) \cong \{\mathbb{Z}, 0, \mathbb{Z} \oplus \mathbb{Z}_{\text{gcd}(N_f, N_c)}, 0, \mathbb{Z} \oplus \mathbb{Z}_{\text{gcd}(N_f, N_c)}, 0, \mathbb{Z}\}$$

- The short exact sequences

$$0 \to \mathcal{C} \to Z_{\widetilde{G}} \to Z_G \to 0 \quad (1)$$

$$0 \to \mathcal{C}^\vee \to \widetilde{\mathcal{A}}^\vee \to \mathcal{A}^\vee \to 0, \quad (2)$$

take the form

$$0 \to \mathbb{Z}_{\text{gcd}(N_f, N_c)} \to \mathbb{Z}_{\text{gcd}(N_f, N_c)} \times \mathbb{Z}_{\text{gcd}(N_f, N_c)} \to \mathbb{Z}_{\text{gcd}(N_f, N_c)} \to 0 \quad (3)$$

$$0 \to \mathbb{Z}_{\text{gcd}(N_f, N_c)} \to \mathbb{Z}_{\text{gcd}(N_f, N_c)} \to 0 \to 0, \quad (4)$$

- Flavor symmetry $G_F = (SU(N_f) \times SU(N_f))\,/Z$ with flavor center $\mathbb{Z}_{\text{gcd}(N_f, N_c)}$

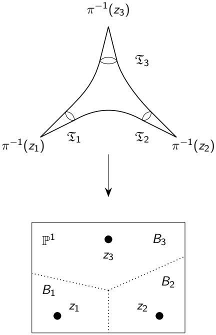
Compact Models: Singular K3 Surfaces

- Singular elliptic K3 surface $\pi : X \to \mathbb{P}^1$
- Singularities of Kodaira type $\Phi_i, \mathfrak{g}_i$ at $z_i \in \mathbb{P}^1$
- Consider patches $B_i \subset \mathbb{P}^1$ centered on z_i
- Define $X^{\text{loc}} = \cup_i \pi^{-1}(B_i)$ and $X^\circ = X \setminus X^{\text{loc}}$
- Mayer-Vietoris sequence $X = X^\circ \cup X^{\text{loc}}$

$$\partial_2 : H_2(X) \;\to\; H_1(\partial X^{\text{loc}})$$

- Relative cycles/Defects of X^{loc} compactify
- Charge lattice enhances [Guralnik, 2001]
- Supergravity gauge group [Apruzzi, Dierigl, Lin, 2020], [Cvetič, Dierigl, Lin, Zhang, 2021]

$$G_{\text{sugra}} = \frac{U(1)^{b_2(X)} \times G_{\text{sc}}}{\text{Im}\,\partial_2}$$



Torus Quotients $X = T^4/\Gamma$

- Isolated singularities with local neighborhood U_i modeled on $\mathbb{C}^2/\mathbb{Z}_{n_i}$
- Define $X^{\mathrm{loc}} = \bigcup_{i=1}^N U_i$ and $X^\circ = X \setminus X^{\mathrm{loc}}$
- Mayer-Vietoris sequence with respect to the covering $X = X^\circ \cup X^{\mathrm{loc}}$

$$0 \to H_2(X^\circ) \oplus \bigoplus_{i=1}^N H_2(U_i) \to H_2(X) \xrightarrow{\partial_2}$$
$$\bigoplus_{i=1}^N H_1(\partial U_i) \to H_1(X^\circ) \oplus \bigoplus_{i=1}^N H_1(U_i) \to H_1(X) \to 0$$

- Resolved Geometry $\widetilde{X} \to X$, exceptional curve lattice L_{E}, K3 lattice L_{K3}

$$0 \to H_2(X^\circ) \oplus L_{\mathrm{E}} \to L_{\mathrm{K3}} \xrightarrow{\partial_2}$$
$$\bigoplus_{i=1}^N H_1(\partial U_i) \to H_1(X^\circ) \to H_1(X) \to 0$$

- Image of ∂_2 partially determined by a lattice embedding problem

Example: Kummer Surface $T^4/\mathbb{Z}_2$

- Singular geometry [Spanier, 1956]:

$$0 \to \mathbb{Z}^6 \to \mathbb{Z}^6 \oplus \mathbb{Z}_2^5 \xrightarrow{\partial_2} \mathbb{Z}_2^{16} \to \mathbb{Z}_2^5 \to 0$$

- Resolved geometry, with the lattice $L_{\mathrm{E}} \cong \mathbb{Z}^{16}$ and $L_{\mathrm{K3}} \cong \mathbb{Z}^{22}$

$$0 \to \mathbb{Z}^6 \oplus L_{\mathrm{E}} \to L_{\mathrm{K3}} \xrightarrow{\partial_2} \mathbb{Z}_2^{16} \to \mathbb{Z}_2^5 \to 0$$

- $\mathrm{Im}\,(\partial_2) = \mathbb{Z}_2^6 \oplus \mathbb{Z}_2^5$, where $\mathbb{Z}_2^5 = L_{\mathrm{Kummer}}/L_{\mathrm{E}}$

(Example Continued)

Local models and affine geometry:

- $\mathbb{Z}_2^{16} = \oplus_{i=1}^{16} H_1(\partial U_i) = \oplus_{i=1}^{16} H_1(\mathbb{RP}^3) \cong \oplus_{i=1}^{16} \langle \mathfrak{T}_i \rangle$
- Characterize $\mathbb{Z}_2^5 \subset \operatorname{Im} \partial_2$ as $(n_I = 0, 1)$ [Barth, Hulek, Peters, Van de Ven, 2004]

$$\text{compact 2-cycle } \sum_{I \in \mathbb{Z}_2^4} n_I \mathfrak{T}_I \quad \Leftrightarrow \quad f : \mathbb{Z}_2^4 \to \mathbb{Z}_2 \,, \ I \mapsto n_I \ \text{ is affine linear}$$

- Supergravity gauge group

$$G_{\rm sugra} = \left(U(1)^6 \times \frac{\left(\bigtimes_{I \in \mathbb{Z}_2^4} SU(2)_I \right)}{\mathbb{Z}_2^5} \right) \Big/ \mathbb{Z}_2^6$$

Conclusion and Outlook

- Kodaira thimbles $\mathfrak{T}$ and center divisors for elliptic CY threefolds X
- Boundary Topology $H_1(\partial X)$, $H_1(\partial X^\circ)$, ...
  $\leftrightarrow$ 0-form, 1-form, 2-group Symmetries
- Cutting and gluing constructions for D6 brane uplifts
  based on Gysin sequence
- Gluing local models to global models
  - Gauged/Broken Higher Symmetries: $T^4/\mathbb{Z}_3$, $T^6/\mathbb{Z}_3$, $T^7/\mathbb{Z}_2^3, \dots$
  - Gauged/Broken 2-Group Symmetries: $T^6/\mathbb{Z}_4$, $T^6/\Gamma, \dots$